

Notes 2: Normal Distribution and Standard Scores

1. Normal Distributions

- bell shaped: symmetrical, unimodal, and smooth (meaning that there is an unlimited number of observations so a polygon or histogram will show a smooth distribution)
- central tendency: mean, median, and mode are identical (note that a normal distribution may have any mean and any variance, e.g., the population distribution of IQs supposedly has $\mu = 100$ and $\sigma = 15$, whereas the population of verbal SAT scores supposedly has $\mu = 500$ and $\sigma = 100$ and both IQ and SAT could be normally distributed)
- standard normal, unit normal curve, z-distribution: a normal distribution with $\mu = 0.00$ and $\sigma = \sigma^2 = 1.00$; any normally distributed data set with scores transformed to z scores will have a standard normal distribution; similarly, any non-normally distributed data with scores transformed to z scores will retain their non-normal shape (z scores do not change shape of distribution)
- area: the area under the normal distribution can be found in a table of z scores, such as the one provided on the next page

2. Relative Position and Standard Scores

- relative position: relative position or relative standing refers to the position of a particular score in the distribution relative to the other scores in the distribution, e.g., percentile ranks, PR; relative position is similar to the concept of norm-referenced interpretations found with many common tests
- standard scores: a standard score in one in which the mean and standard deviation of the distribution will have pre-defined values, e.g., z scores have a M of 0.00 and s of 1.00; note that PR are not standard scores
- z score: z scores provide a measure of the relative standing for a score within a distribution based upon two things, the mean and standard deviation; z scores are standard scores since $M = 0.00$ and $s = 1.00$; the formula for a sample is:

$$z = \frac{X - \bar{X}}{SD} = \frac{X - \bar{X}}{s}$$

- interpretation of z scores: since z's are expressed in standard deviations unit from the mean, a z of 1 implies that the raw score is 1 standard deviation above the mean of the distribution; if z is -2, then the corresponding raw score is 2 standard deviations below the group mean; a z of -1.34 indicates that the raw score is 1.34 standard deviations below the mean of the group
- mean, variance of z: the mean of z scores is 0.00, and the standard deviation (and variance) of z scores is 1.00.
- shape of original distribution: if one transforms a set of raw scores into z scores, the transformation does not change the shape of the original distribution; if the original distribution was normally distributed, then the z score distribution will remain normally distributed; if the original distribution was bimodal, then the z score distribution will be bimodal—transforming Xs into z scores changes only the mean and variance of the distribution, with the transformed mean equal to 0.00 and the transformed variance equal to 1.00

Example Calculation of Z scores

Raw Scores	Mean	Deviation Scores	Deviation Scores Squared	z-scores
X_i	\bar{X}	$X - \bar{X}$	$(X - \bar{X})^2$	z
99	101.2	-2.2	4.84	-0.261
112	101.2	10.8	116.64	1.280
89	101.2	-12.2	148.84	-1.446
105	101.2	3.8	14.44	0.450
101	101.2	-0.2	0.04	-0.024

Raw scores = X_i

Mean = $\sum X/n = 506/5 = 101.2$

Sums of Squares (SS) = $\sum (X - \bar{X})^2 = 284.8$

Variance = $s^2 = \frac{\sum (X - \bar{X})^2}{n-1} = \frac{SS}{n-1} = \frac{284.8}{4} = 71.2$

Standard Deviation = $s = \sqrt{s^2} = \sqrt{\frac{\sum (X - \bar{X})^2}{n-1}} = \sqrt{71.2} = 8.438$

z-score = $z = \frac{X - \bar{X}}{SD} = \frac{X - \bar{X}}{s}$

3. Finding area under the normal curve

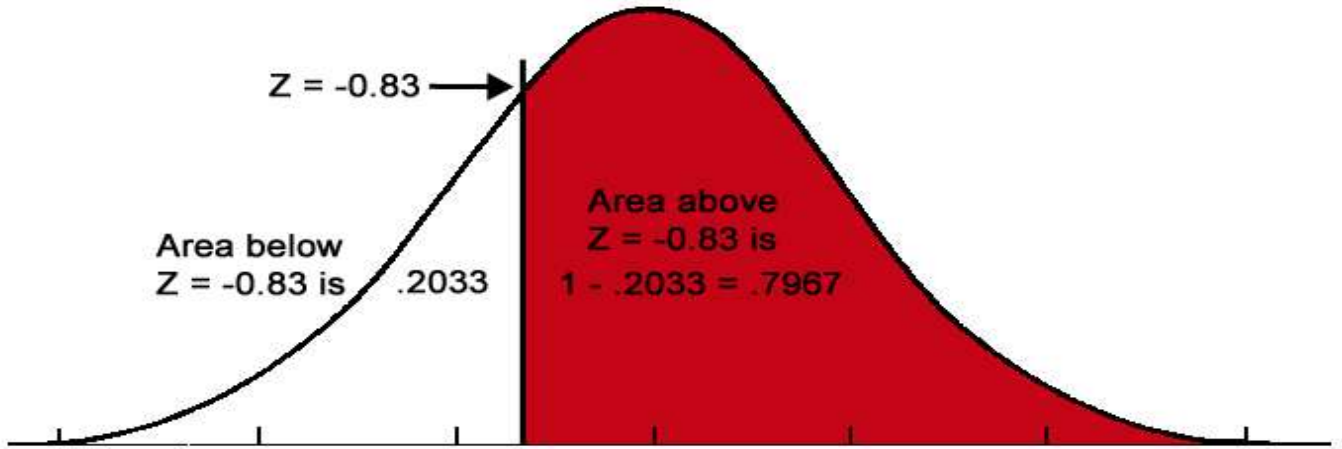
To find area under the normal distribution, one must first convert raw scores, X , to Z score, then consult a Z table showing area under the normal curve.

A Z table is provided at the end of this document and also separately in the course material.

The Z table provided shows the area to the left of a Z score, whether the Z is negative or positive. For example, to find the area to the left of a negative Z, such as $Z = -0.83$, the table shows the following (highlighted in red): .2033

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-1.00	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
-0.90	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
-0.80	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-0.70	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
-0.60	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451

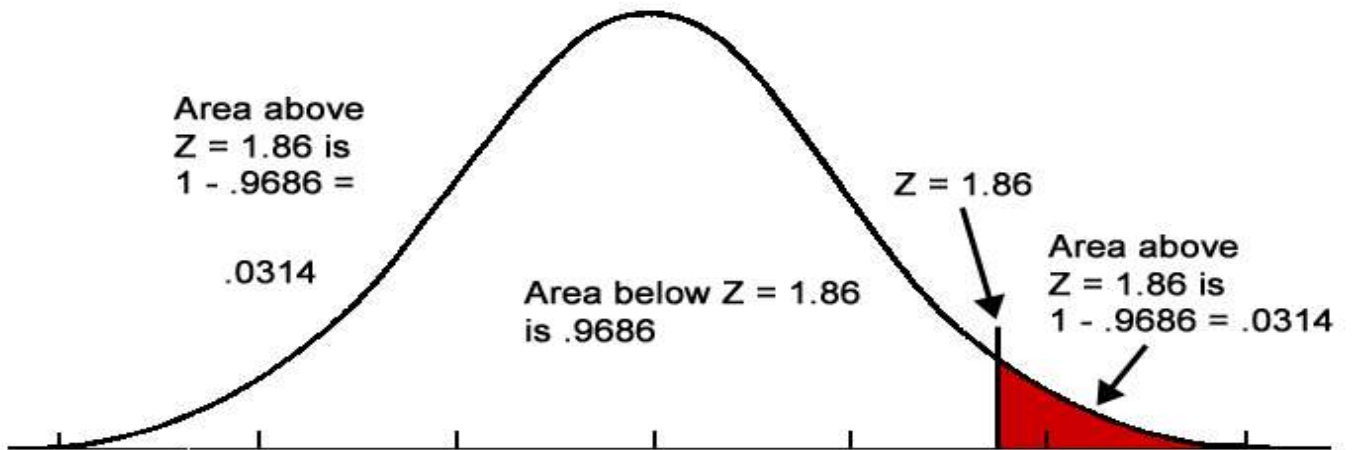
The graphic below shows the area, or proportion of scores, below $Z = -0.83$ and the proportion above a Z of -0.83 .



Similarly, to find the area for a positive Z score, use the second part of the provided Z table. For example, to find the area below $Z = 1.86$, locate Z of 1.86 and the intersection represents the proportion of scores below that Z score. See example below highlighted in red.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.60	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.70	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.80	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.90	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.00	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817

The graphic below shows the area for a $Z = 1.86$.



Recall that the formula for converting raw scores to z scores is

$$z = \frac{X - \bar{X}}{s}$$

so if one wished to determine the percentage of IQ scores that were less than 110, the following calculation is used (note that the mean of IQ is 100 and standard deviation is 15):

$$z = \frac{X - \bar{X}}{s} = \frac{IQ - \bar{IQ}}{s_{IQ}} = \frac{110 - 100}{15} = 10/15 = 0.67$$

and the proportion of scores below $Z = 0.67$ is .7486 or 74.86%.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.50	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.60	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.70	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852

Exercise 1

Determine the proportion of scores that:

- less than or equal to 1.25?
- less than or equal to -.55?
- greater than or equal to 1.94?
- greater than or equal to -1.23?
- fall between the mean ($z = 0.00$) and 0.33?
- lie between -0.75 and -1.93?
- greater than 1.96 or less than -1.96?
- greater than 2.58 or less than -2.58?

Answers to Exercise 1

- | | |
|---|--|
| 1. less than or equal to 1.25? | $p = (Z \leq 1.25) = .8944$ |
| 2. less than or equal to -.55? | $p = (Z \leq -0.55) = .2912$ |
| 3. greater than or equal to 1.94? | $p = (Z \geq 1.94) = 1 - .9738 = .0262$ |
| 4. greater than or equal to -1.23? | $p = (Z \geq -1.23) = 1 - .1093 = .8907$ |
| 5. fall between the mean ($z = 0.00$) and 0.33? | $p = (0.00 \leq Z \leq 0.33) = .6293 - .5000 = .1293$ |
| 6. lie between -0.75 and -1.93? | $p = (-0.75 \leq Z \leq -1.93) = .2266 - .0268 = .1998$ |
| 7. greater than 1.96 or less than -1.96? | $p = (Z \leq -1.96 \text{ or } Z \geq 1.96) = (Z \geq 1.96) = .05$ |
| 8. greater than 2.58 or less than -2.58? | $p = (Z \leq -2.58 \text{ or } Z \geq 2.58) = (Z \geq 2.58) = .0098 \approx .01$ |

4. Xs to z ($X \rightarrow z$), finding area under normal curve

Remember that converting a distribution of raw scores to z scores will not change the shape of the original distribution, so if the raw score distribution is not normal, then the corresponding z score distribution will not be normal.

For raw scores that are normally distributed, one may convert raw scores into z scores and then find the proportion of scores above or below certain raw scores.

Exercise 2

Assuming that SAT scores are normally distributed with a $M = 500$ and $SD = 100$, provide answers to the following:

- find the proportion of scores between 655 and the mean;
- find the proportion of scores between 459 and the mean;
- find the proportion of scores below 645;

4. find the proportion of scores below 468;
5. find the proportion of scores above 645;
6. find the proportion of scores above 468;
7. find the proportion of scores above 500;
8. find the proportion of scores between 720 and 755;

Answers to Exercise 2

1. find the proportion of scores between 655 and the mean;

$Z = (655-500)/100 = 1.55$ and mean is 500 so $Z = (500-500)/100 = 0.00$, so find area between 1.55 and 0.00.

$p = .9394 - .5000 = .4394$

2. find the proportion of scores between 459 and the mean;

$Z = (459-500)/100 = -0.41$ and mean is 500 so $Z = (500-500)/100 = 0.00$, so find area between -0.41 and 0.00.

$p = .5000 - .3409 = .1591$

3. find the proportion of scores below 645;

$Z = (645-500)/100 = 1.45$, so area below 1.45 is $p = .9265$

4. find the proportion of scores below 468;

$Z = (468-500)/100 = -0.32$, so area below -0.32 is $p = .3744$

5. find the proportion of scores above 645;

$Z = (645-500)/100 = 1.45$ so area above 1.45 is $p = 1 - .9265 = .0735$

6. find the proportion of scores above 468;

$Z = (468-500)/100 = -0.32$ so area above -0.32 is $p = 1 - .3745 = .6255$

7. find the proportion of scores above 500;

$Z = (500-500)/100 = 0.00$ and area above (and below) is $p = .50$

8. find the proportion of scores between 720 and 755;

$Z = (720-500)/100 = 2.2$ and $Z = (755-500)/100 = 2.55$, so find area between 2.2 and 2.55.

$p = .9946 - .9861 = .0085$

5. Percentile Rank (PR), $z \rightarrow PR$

Recall that a percentile rank (PR) indicates the percentage of scores that lie below a certain score in a distribution. Like a Z score PR indicates where a score is located in relation to other scores.

There are two methods for determining PR:

- The first was introduced with frequency distributions. The cumulative relative frequency is a PR measure and is useful for distributions that are not normal in shape.
- The second is based upon the Z table and is useful when data follow a normal distribution. The percentage of scores below a given Z value is the PR.

Thus, using the second approach, if a set of scores is normally distributed, you may find the percentile rank using a z table after converting the X to z. The formula for PR is:

$$PR = (\text{proportion below } z)100$$

For example, IQ is thought to have a population mean, μ , of 100 with a population SD, σ , of 15. What is the PR of someone with an IQ of 115?

First, convert 115 to a Z score: $Z = (115-100)/15 = 15/15 = 1$

Second, find area (proportion) under $Z = 1$ in normal distribution: $p = .8413$

Third, convert this proportion to PR: $PR = (.8413)*100 = 84.13$

Exercise 3

Determine the appropriate PRs.

1. What is the PR for $z = -1.33$?
2. What is the PR for $z = 2.20$?
3. What is the PR for $z = 0.00$?
4. What is the PR for $z = -2.56$?

Answers to Exercise 3

1. What is the PR for $z = -1.33$? **$p = .0918$ so $PR = 9.18$**
2. What is the PR for $z = 2.20$? **$p = .9861$ so $PR = 98.61$**
3. What is the PR for $z = 0.00$? **$p = .5000$ so $PR = 50.00$**
4. What is the PR for $z = -2.56$? **$p = .0052$ so $PR = 0.52$**

Percentile Ranks (PR) and Raw Score Changes

Note that PRs tend to change quickly for small changes in raw scores near the middle of a distribution of raw scores that are normal in shape, but changes less quickly for raw scores that are further from the mean. Thus, a difference of one PR, say from 50 to 51 or from 90 to 91, does not indicate the same amount of change in raw score units when raw scores take a normal shape.

For example, the difference in IQ units for PRs of 50 to 55 is 100 to 102—only two IQ points. However, as one moves further from the IQ mean, the difference of 5 percentile rank points represents a greater difference in IQ points. For instance, a change in PR from 90 to 95 corresponds to a change of IQ points from 119 to 125—a 6 point difference. In short, PR and raw scores do not change in a linear fashion.

6. Convert Z scores to Raw Scores X

To convert from Z to X use the following formula:

$$X = M + Z \times SD$$

For example, using IQ scores with $M = 100$ and $SD = 15$, Z scores of -1.13, -2.5, and 1.89 yield the following IQ scores:

$$Z = -1.13, \text{ so}$$

$$X = 100 + -1.13 \times 15 = 100 - 16.95 = 83.05$$

$$Z = -2.50, \text{ so}$$

$$X = 100 + -2.50 \times 15 = 100 - 37.5 = 62.5$$

$$Z = 1.89, \text{ so}$$

$$X = 100 + 1.89 \times 15 = 100 + 28.35 = 128.35$$

7. Skewed Distributions

- skewness: the degree of asymmetry of a distribution from a normal distribution; the greater the skew of a distribution, the greater the difference (in standardized units) between the three primary measures of central tendency.
- positive skew: this type of distribution has an elongated tail in the positive direction on a number line; the order of the measures of central tendency, from smallest to largest, is mode, median, and mean
- negative skew: this type of distribution has an elongated tail in the negative direction on a number line; the order of the measures of central tendency, from smallest to largest, is mean, median, and mode

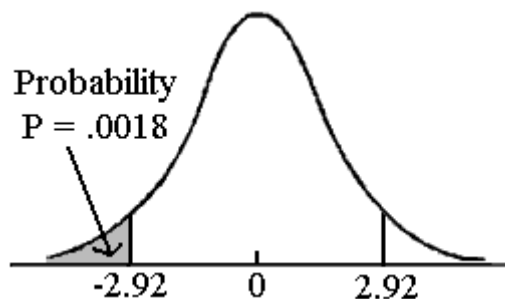
8. Kurtosis

This refers to the peakedness of a distribution; there are three types of kurtosis:

- * platykurtic: flat or broad distributions; more extreme scores than normal
- * mesokurtic: normal distributions
- * leptokurtic: slender or narrow, more peaked in appearance; fewer extreme scores than normal

Z Table: Negative Values(Source of table: <http://www.stat.tamu.edu/stat30x/zttables.php>)

Body of table gives area under Z curve to the left of z.

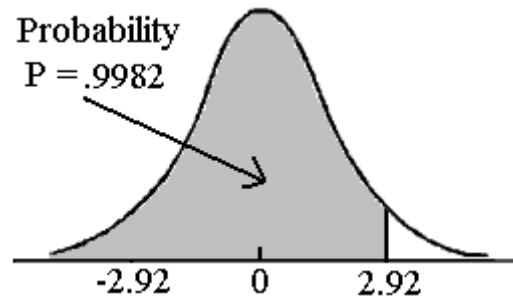
Example: $P[Z < -2.92] = .0018$ 

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.80	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
-3.70	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
-3.60	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
-3.50	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
-3.40	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
-3.30	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-3.20	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
-3.10	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
-3.00	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.90	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.80	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.70	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
-2.60	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.50	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
-2.40	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.30	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.20	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.10	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
-2.00	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.90	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
-1.80	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
-1.70	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
-1.60	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
-1.50	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
-1.40	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
-1.30	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.20	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
-1.10	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
-1.00	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
-0.90	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
-0.80	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-0.70	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
-0.60	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
-0.50	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
-0.40	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
-0.30	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
-0.20	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
-0.10	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
-0.00	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Z Table: Positive Values

Body of table gives area under Z curve to the left of z .

Example: $P[Z < 2.92] = .9982$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.00	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.10	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.20	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.30	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.40	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.50	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.60	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.70	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.80	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.90	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.00	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.10	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.20	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.30	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.40	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.50	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.60	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.70	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.80	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.90	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.00	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.10	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.20	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.30	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.40	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.50	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.60	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.70	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.80	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.90	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.00	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.10	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.20	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.30	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.40	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.50	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998
3.60	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.70	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.80	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999